THE RAMSEY NUMBER AND SATURATION OF THE TRISTAR

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1. Context

In this paper, we provide results on the Ramsey number and saturation of diameter four caterpillars. We refer to such trees as tristars. This work began as a joint project with Spencer Brooks during the 2011 Graphs, Groups, and Geometry REU at the University of North Carolina-Asheville. We obtained the exact Ramsey number for a particular subfamily of tristars called fountains and showed that fountains are Ramsey unsaturated. Additionally, we calculated bounds on the Ramsey number of "regular tristars". The work continued at Clemson University with Emily Nystrom as we computed the bounds for more general tristars, and in some cases, the bounds are slightly tighter than currently known ones.

Consider simple, undirected, and connected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Recall that the *Ramsey number of* G_1 *versus* $G_2, r(G_1, G_2)$, is the smallest integer n so that every two coloring (say, with red and blue) of the edges of K_n contains either a monochromatic G_1 or G_2 as an edge-induced isomorphic subgraph. When $G_1 = G_2$, we write $r(G_1)$. A graph G is *Ramsey saturated,* or simply *saturated*, if for every edge $e \notin E(G), r(G + e) > r(G)$. Otherwise, G is *unsaturated*.

The *caterpillar* $C(m_1, \ldots, m_k)$ consists of a path on $k \geq 1$ vertices with m_i leaves adjacent to the i^{th} vertex of the path. We assume $m_1, m_k \neq 0$. For $k = 1, C(m)$ is a star and it is known [3] that

$$r(C(m)) = \begin{cases} 2m - 1 & m \text{ even} \\ 2m & m \text{ odd.} \end{cases}$$

Though it is conjectured almost all graphs and all non-star trees of order at least 5 are unsaturated, stars are saturated [2]. For $k = 2$, we write $B(m, n)$ and call the graph a bistar. In [1], we see $r(B(m, n)) \geq 2m + n + 2$ for $n \geq m$, and equality holds when $m \geq 2$ and $n \in \{m, m + 1\}$. They showed that in general

$$r(C(m_1, \ldots, m_k)) \geq |V| + n - 1$$

where

$$n_1 = \sum_{i=1}^{\lceil k/2 \rceil} m_{2i-1} + \lfloor k/2 \rfloor \quad n_2 = \sum_{i=1}^{\lfloor k/2 \rfloor} m_{2i} + \lceil k/2 \rceil$$

$$n = \min\{n_1, n_2\}.$$

Moreover, they proved all bistars are unsaturated.

Our work continues this progression by examining $k = 4$, the tristars. For $v \in V$, let

$$B(v) = \{u \in V : uv \in E \text{ blue}\}$$

be the set of blue neighbors and $d_B(v) = |B(v)|$ be the blue degree of v. The set of red neighbors and the red degree, respectively, $R(v)$ and $d_R(v)$, are similarly defined. We begin by establishing bounds on $r(C(a, b, c))$. Without loss of generality, assume $c \geq a$ and $a, b, c \geq 1$.

2. Bounds on the Ramsey number of tristars

Perhaps the best lower bound on the Ramsey number of a tree T originates from viewing the tree as a bipartite graph with parts t_1 and $t_2, t_2 \geq t_1$ [5]. Then $r(T) \geq \max\{2t_1 + t_2 - 1, 2t_2 - 1\}$. Applying this to the tristars yields

$$r(C(a,b,c)) \geq \begin{cases} 2(a+c)+b+3 & \text{if } 2(a+c) > b \\ 2b+3 & \text{if } 2(a+c) \leq b \\ a+c+2b+4 & \text{if } 3 \leq a+c < 2b+3 \\ 2a+2c+1 & \text{if } 2b+3 \leq a+c. \end{cases} \tag{1}$$

We slightly improve this bound for trees containing a single vertex of high odd degree relative to the other vertices.

Proposition 2.1. *Let G be any graph such that the maximum degree $\Delta = \Delta(G) \geq 3$ and odd. Then $R(G) \geq 2\Delta$.*

Proof. Consider $K_{2\Delta-1}$. For each vertex $v_i \in \{v_0, v_1, \ldots, v_{2\Delta-2}\}$ of $K_{2\Delta-1}$ and integer $j \in \left\{1, \ldots, \frac{\Delta-1}{2}\right\}$, color every edge $v_i v_k$ blue, where $k = (i+j) \mod (2\Delta - 1)$. Color the remaining edges red. Notice that for each $v_i, d_B(v_i), d_R(v_i) = \Delta - 1$. ☺

This is a sort of converse to Theorem 9.2.2 (an Erdös conjecture) in [4]:

Theorem 2.2. *(Chvátal, Rödl, Szemerédi, and Trotter) For every positive integer k, there exists a constant c so that*

$$r(H) \leq c|H|$$

for all graphs H satisfying $\Delta(H) \leq k$.

In the case when $2(a+c) \leq b$ for odd b, Proposition 2.1 gives $r(C(a,b,c)) \geq 2b+4$.

For an upper bound on the Ramsey number of tristars, we want to use the probabilistic method, but there is no clear way to count the number of possible tristars appearing in a complete graph. We plan on developing a way to do this at a later time. An adequate upper bound using a constructive approach is to set $m = \max\{a,b,c\}$ and observe that $r(C_3^m) := r(C(m,m,m)) \geq r(C(a,b,c))$. Hence, bounding the Ramsey number of the m-regular tristar C_3^m bounds the Ramsey number for all tristars.

Proposition 2.3.

$$r(C_3^m) \leq 5m + 4.$$

Proof. Omitted. ☺

3. Ramsey number and saturation of fountains

Denote the fountains by $F_b := C(1,b,1)$. Call the vertex of degree $b+2$ the center vertex v_c, the vertices of degree 2 the branches b_1, b_2, and label the leaves ℓ_i beginning with b_1 and moving across to b_2. See Figure 1 for an example with F_3. Using Equation 1, Proposition 2.1, and some simple observations for $b = 1, 2$, we obtain

$$r(F_b) \geq \begin{cases} 8 & b = 1 \\ 9 & b = 2 \\ 2b+3 & \text{even } b \geq 4 \\ 2b+4 & \text{odd } b \geq 3. \end{cases} \tag{2}$$

It turns out that this lower bound is precisely the Ramsey number of the fountain. To prove this, we need the following results.

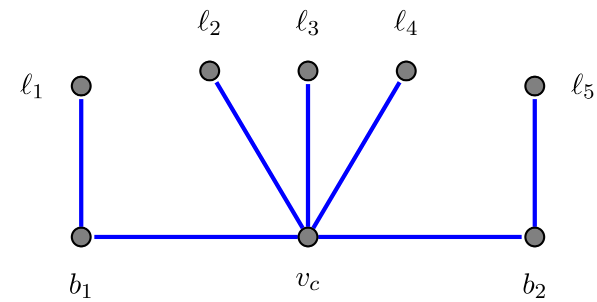

FIGURE 1. We use these labels to identify the fountain in our proofs.

Lemma 3.1. *If $n \geq 5$ and is an odd integer, then there exists $v \in V(K_n)$ such that $d_B(v)$ and $d_R(v)$ are even.*

Proof. We proceed by contradiction. Suppose there exists a two-coloring of K_n such that all vertices $v \in V(K_n)$ have an odd blue degree. Then the number of blue edges is $\frac{\sum_{i=1}^{n} d_B(v_i)}{2}$, which is not an integer. Thus, $d_B(v)$ must be even for some $v \in V(K_n)$. Note that if $d_B(v)$ is even, then so is $d_R(v)$. ☺

In conjunction with the Pigeonhole Principle, Lemma 3.1 implies

Corollary 3.2. *Suppose $b \geq 4$ and is even. Then there exists a vertex $v \in V(K_{2b+3})$ such that $d_B(v) \geq b+2$ and $d_R(v) \leq b$.*

We now prove the upper bound.

Proposition 3.3.

$$r(F_b) \leq \begin{cases} 8 & b = 1 \\ 9 & b = 2 \\ 2b+3 & \text{even } b \geq 4 \\ 2b+4 & \text{odd } b \geq 3. \end{cases}$$

Proof. Here, we only prove the even case when $b \geq 4$. The odd and the $b = 1, b = 2$ cases are similar to the following proof. By Corollary 3.2, there exists a $v_0 \in V(K_{2b+3})$ such that $d_B(v_0) \geq b+2$ and $d_R(v_0) \leq b$. Let $T \subseteq B(v_0)$ such that $|T| = b+2$, and let $S = V(K_{2b+3}) \setminus (T \cup \{v_0\})$. To avoid a blue F_b, we observe that for any two vertices $t_1, t_2 \in T$ and $s_1, s_2 \in S$ where $t_1 \neq t_2$ and $s_1 \neq s_2$, the edges $t_1 s_1, t_2 s_2$ cannot both be blue; otherwise, v_0 serves as v_c since it has the sufficient number of blue neighbors, t_1 and t_2 are the branches, and s_1, s_2 are ℓ_1, ℓ_{b+2}. For this reason, it must be the case either that $s_1 = s_2$, that $t_1 = t_2$, or that no blue edges are between S and T:

(1) Exactly one vertex $s_1 \in S$ satisfies $|B(s_1) \cap T| \geq 1$.
(2) Exactly one vertex $t_1 \in T$ satisfies $|B(t_1) \cap S| \geq 1$.
(3) For all vertices $t \in T, |B(t) \cap S| = 0$.

Case 1: Notice that for all $s \in S \setminus \{s_1\}, |R(s) \cap T| = b+2$ creating a red fountain. Select any vertex $s_i \in S \setminus \{s_1\}$ to be v_c, and since $|R(s_i) \cap T| = b+2$, we have sufficient leaves and branches. Each $t \in T$ has $|R(t) \cap S| \geq b-1$ so choose appropriate vertices, distinct from s_i and s_1, for ℓ_1, ℓ_{b+2}.

Case 2: Notice that because $|R(s_i) \cap T| \geq b+1$ for all vertices $s_i \in S$, no vertex $s_1 \in S$ can share a red edge with v_0 or with another vertex $s_2 \in S$. Indeed, if such an edge was present between s_1 and s_2, then s_1 serves as v_c, and s_2 along with s_1's $b-1$ red neighbors in T are leaves. The remaining 2 vertices $t_i \in (R(s_1) \cap T)$ are the brances, and because each $t \in T \setminus \{t_1\}$ has $|R(t) \cap S \setminus \{s_1, s_2\}| = b-2$, select any 2 to be ℓ_1, ℓ_{b+2}. An analogous argument also reveals a red fountain if a red edge is present between v_0 and some vertex $s \in S$.

Thus, those edges must all be blue; yet, this creates a blue fountain. With this coloring, let v_0 be v_c. Select t_1 to serve as b_1 and a vertex $s_1 \in (B(t_1) \cap S)$ as ℓ_1. Now, pick a vertex $s_2 \in S \setminus \{s_1\}$ to be b_2 and another vertex $s_3 \in S \setminus \{s_1\}$ as its leaf. To complete the fountain, let $b-1$ leaves be in $T \setminus \{t_1\}$ and the remaining leaf be in $S \setminus \{s_1, s_2, s_3\}$.

Case 3: Unsurprisingly, this creates a red fountain in exactly the same manner as Case 1. ☺

Combining Equation 2 and Proposition 3.3, we get

Theorem 3.4.

$$r(F_b) = \begin{cases} 8 & b = 1 \\ 9 & b = 2 \\ 2b+3 & \text{even } b \geq 4 \\ 2b+4 & \text{odd } b \geq 3. \end{cases}$$

Now, we address saturation.

Theorem 3.5. *For $b \geq 1, F_b$ is unsaturated.*

Proof. To show that F_b is unsaturated, we demonstrate that there is an edge $e \notin E(F_b)$, such that any two-coloring of $K_{r(F_b)}$ results in a monochromatic copy of $F_b + e$. We use the edge between a leaf and a branch vertex.

Since the proof for the even and odd cases are completely analogous, we prove only the even case. Suppose b is even. Recall the cases considered in the proof of Proposition 3.3. In Case 1 and 3, there is always a red edge $e \notin E(F_b)$ between a leaf in T and a branch vertex in S. For Case 2, a blue edge $e \notin E(F_b)$ connects a leaf and branch vertex, which both lie in S. Thus, $r(F_b) = r(F_b+e)$ implying F_b is unsaturated for even b. ☺

4. Future Directions

Our next step is to tighten the bounds on Ramsey number for general tristars. For the m-regular case, we conjecture $r(C_3^m) = 4m+4$. Once we do this, we want to see if we can generalize, perhaps through the use of the probabilistic method, to bounds on $r(C(m_1, \ldots, m_k))$ for any k.

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