THE LOCATION OF ROOTS OF LOGARITHMICALLY CONCAVE POLYNOMIALS

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Abstract. We prove an elementary but useful relationship between the logarithmic concavity of a polynomial with positive coefficients and the location of that polynomial's roots in the complex plane.

1. Introduction

Our goal in this note is to connection the coefficients of a polynomial with positive coefficients to the location of that polynomial's roots in the plane. Loosely speaking, such a study is not a new one, as a number of authors have considered the problem of locating the roots of a given polynomial. Many studies take as their starting point the following classical (and easy-to-prove) theorem, due to Gershgorin, in [2]:

Theorem 1.1. *Let $A = [a_{ij}]$ be an $n \times n$ complex-valued matrix and for each $i = 1, ..., n$ let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the moduli of the off-diagonal elements of the ith row of A. Then each eigenvalue of A lies in the union of the circles defined by $|z - a_i i| \leq R_i$ $(i = 1, ..., n)$. An analogous result holds for the columns of A.*

Given a polynomial p it is easy to construct the corresponding *companion matrix*, whose eigenvalues are precisely the roots of A. Thus Theorem 1.1 can be applied to give estimates for the roots of a polynomial, placing them in the union of a collection of closed disks in $\mathbb{C}$. Older results along these lines are due to Wilf [6] and Bell [1]; more recently Zamfir [7] provides similar estimates.

All of the above results provide estimates on the moduli of the roots of a given polynomial. However, in [4], Handelman presents a method for finding a region of the complex plane containing all of the roots of a real polynomial $p = \sum_{i=0}^{n} c_i x^i$ with non-negative coefficients, c_i. The form this region takes depends on the value of $\beta(p) = \inf_{i=1}^{n-1} \frac{c_i^2}{c_{i-1}c_{i+1}}$. Handelman's work extends work of Kurtz [5], in which conditions are founding guaranteeing the roots of a polynomial will be real. Our main result (Theorem 1.2 below) provides a sort of converse to these last results, enabling us to derive information about a polynomial's coefficients provided its roots lie in a particular region in $\mathbb{C}$.

Recall that a sequence $\{k_i\}_{i=0}^{n}$ of nonnegative numbers is said to be *logarithmically concave* (or simply *log concave*) if for every $i = 1, ..., n-1$ the inequality $k_i^2 \geq k_{i-1}k_{i+1}$ holds. A polynomial $p(z) = \sum_{i=0}^{n} k_i z^i$ with nonnegative coefficients is called log concave if its coefficient sequence is log concave. It is easy to show that a log concave polynomial is also *unimodal*: there exists an index j such that

$$k_0 \leq k_1 \leq \cdots \leq k_{j-1} \leq k_j \geq k_{j+1} \geq \cdots \geq k_{n-1} \geq k_n.$$

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Our main result is as follows:

Theorem 1.2. *Let* $p(z) = \sum_{i=0}^{n} k_i z^i$ *be a polynomial with positive coefficients. If every root* $z = a + bi$ *of* p *satisfies both* $a \leq 0$ *and* $b^2 \leq 3a^2$*, then* p *is log concave.*

Restated in terms of polar coordinates, Theorem 1.2 says that if every root lies in the sector bounded by the rays $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$, then p is log concave. The flavor of this result is similar to that of the following well-known theorem, appearing in [3]:

Theorem 1.3. *Let* $p(z) = \sum_{i=0}^{n} k_i z^i$ *be a polynomial with all real roots. Then the sequence* $\left(k_i / \binom{n}{i}\right)$ *is log concave.*

Clearly the conclusion of our result is not as strong as that of Theorem 1.3, but the hypothesis on the location of the roots of p is considerably weaker. Our proof, contained in the following section, is completely elementary and requires only basic knowledge of complex arithmetic.

2. Proof of the main theorem

We prove Theorem 1.2 by induction on the degree, d, of p. To begin, note that every root $z = a + bi$ satisfies $a \leq 0$. We will also rely on the following lemma, which follows almost trivially from the definition of log-concavity:

Lemma 2.1. *Suppose the sequence* $\{k_i\}_{i=0}^{n}$ *is log concave. Then for all* i, j *satisfying* $0 < i \leq j < n$, $k_i k_j \geq k_{i-1} k_{j+1}$.

We now proceed with our proof. To make our work slightly easier we note that we may assume that p is monic, for dividing every term through by a positive leading coefficient will affect neither the location of p's roots nor p's log concavity. The first nontrivial case is $d = 2$. If p factors as $(z - r)(z - s)$ for $r, s \in \mathbb{R}$, then $p(z) = z^2 - (r + s)z + rs$ is log concave if and only if $(r + s)^2 \geq rs$, which reduces to $r^2 + s^2 \geq 3rs$. Since $r, s \leq 0$, this is trivially true. If the roots of p appear as a conjugate pair $a \pm bi$, then $p(z) = z^2 - 2az + a^2 + b^2$, which is log concave if and only if $3a^2 \geq b^2$, which is precisely our hypothesis on the location of the roots z.

Now suppose we have proven our result for polynomials of degree at most n, and let $\deg(p) = n+1$ be a polynomial whose roots satisfy the hypothesis of Theorem 1.2.

First consider the factorization $p(z) = (z - r)q(z)$ for r real, and let k_i represent the coefficient on x^i in q. Note that $r \leq 0$. Multiplying out the righthand side and collecting powers, we find that log-concavity obtains when

(1) $(k_0 - k_1 r)^2 \geq -k_0 r(k_1 - k_2 r)$,
(2) $(k_i - k_{i+1} r)^2 \geq (k_{i-1} - k_i r)(k_{i+1} - k_{i+2} r)$ for $i = 1, ..., n - 2$, and
(3) $(k_{n-1} - r)^2 \geq k_{n-2} - k_{n-1} r$.

Expanding (1) we obtain the inequality $k_0^2 - k_0 k_1 r + k_1^2 r^2 \geq k_0 k_2 r^2$. Note that $-k_0 k_1 r \geq 0$, and by log-concavity of q, $k_1^2 \geq k_0 k_2$. Thus the inequality holds. (3) follows in a similar fashion, using the fact that $-k_{n-1} r \geq 0$, and that $k_{n-1}^2 \geq k_{n-2} = k_{n-2} k_n$.

Expanding (2) we obtain

$$k_i^2 - 2k_i k_{i+1} r + k_{i+1}^2 r^2 \geq k_{i-1} k_{i+1} - k_{i-1} k_{i+2} r - k_i k_{i+1} r + k_i k_{i+2} r^2.$$

By log concavity of q, it suffices to prove the inequality

$$2k_i k_{i+1} \geq k_{i-1} k_{i+2} + k_i k_{i+1},$$

obtained by comparing the first and last terms on both sides and factoring out $-r \geq 0$. But note that this follows from Lemma 2.1.

Now we consider the factorization $p(z) = (z - z_0)(z - \overline{z_0})q(z)$. Suppose $z_0 = a + bi$; thus $p(z) = (z^2 - 2az + a^2 + b^2)q(z)$. Expanding p as above, we again obtain several cases, examining each consecutive triple of p's coefficients in turn. Let $p(z) = \sum \ell_i x^i$.

Case 1: $\ell_{n+1}^2 \geq \ell_n \ell_{n+2}$**.** Expanding the relevant coefficients in terms of the coefficients k_i and collecting like terms whenever possible we see we must prove the inequality

$$3a^2 + k_{n-1}^2 \geq b^2 + k_{n-2} + 2ak_{n-1}.$$

The last term on the righthand side is negative, so removing it will yield a stronger inequality. Note also that log-concavity of q implies $k_{n-1}^2 \geq k_n k_{n-2} = k_{n-2}$, so we may compare and discard these terms and yield an even stronger inequality, namely $3a^2 \geq b^2$. But this is simply the hypothesis of Theorem 1.2, so the inequality we desire holds in this case.

Case 2: $\ell_n^2 \geq \ell_{n-1}\ell_{n+1}$**.** Expanding and collecting like terms, we must show

$$\begin{aligned} & k_{n-1}^2 + 3a^2 k_{n-1}^2 + 2b^2 k_{n-2} + (a^2 + b^2)^2 \\ \geq\ & k_{n-2}k_{n-3} + b^2 k_{n-1}^2 + 2a^2 k_{n-2} + 2ak_{n-1}k_{n-2} - 2ak_{n-3} + 2a(a^2 + b^2)k_{n-1}. \end{aligned}$$

Since the last term on the lefthand side is positive and the last term on the righthand side is negative, the above follows from the stronger inequality obtained by removing these terms. Moreover, log-concavity of q implies that $k_{n-2}^2 \geq k_{n-1}k_{n-3}$, so we may remove the first term on either side and reduce our problem to proving

$$3a^2 k_{n-1}^2 + 2b^2 k_{n-2} \geq b^2 k_{n-1}^2 + 2a^2 k_{n-2} + 2ak_{n-1}k_{n-2} - 2ak_{n-3}.$$

The last two terms on the righthand side combine to give $2a(k_{n-1}k_{n-2} - k_n k_{n-3})$, which is negative, by Lemma 2.1. Thus this term can be removed as well to obtain the stronger inequality

$$3a^2 k_{n-1}^2 + 2b^2 k_{n-2} \geq b^2 k_{n-1}^2 + 2a^2 k_{n-2}.$$

If $|a| \leq |b|$, then our hypothesis $b^2 \leq 3a^2$ implies the last inequality immediately, comparing the corresponding terms on either side. Suppose then that $|a| > |b|$. Then

$$3a^2 k_{n-1}^2 = a^2 k_{n-1}^2 + 2a^2 k_{n-1}^2 \geq b^2 k_{n-1}^2 + 2a^2 k_{n-2},$$

where the inequality follows by comparing the first terms directly and using log-concavity on the second terms.

Case 3: $\ell_i^2 \geq \ell_{i-1}\ell_{i+1}$**,** $i = 3, ..., n-1$**.** The coefficients in this case possess the general form

$$\ell_i = k_{i-2} - 2ak_{i-1} + (a^2 + b^2)k_i.$$

After expanding and combining like terms, we must show

$$\begin{aligned} & k_{i-2}^2 - 2ak_{i-1}k_{i-2} + 2(-a^2 + b^2)k_i k_{i-2} + (3a^2 - b^2)k_{i-1}^2 - 2a(a^2 + b^2)k_i k_{i-1} + (a^2 + b^2)^2 k_i^2 \\ \geq\ & k_{i-1}k_{i-3} - 2ak_i k_{i-3} + (a^2 + b^2)k_{i+1}k_{i-3} - 2a(a^2 + b^2)k_{i+1}k_{i-2} + (a^2 + b^2)^2 k_{i+1}k_{i-1}. \end{aligned}$$

We begin by noting that our desired inequality follows from

$$\begin{aligned} & -2ak_{i-1}k_{i-2} + 2(-a^2 + b^2)k_i k_{i-2} + (3a^2 - b^2)k_{i-1}^2 - 2a(a^2 + b^2)k_i k_{i-1} \\ \geq\ & -2ak_i k_{i-3} + (a^2 + b^2)k_{i+1}k_{i-3} - 2a(a^2 + b^2)k_{i+1}k_{i-2}. \end{aligned}$$

To see this, we compare the first and last terms on the left-hand side of the desired inequality with the first and last terms on the right. By the log concavity of q, $k_{i-2}^2 \geq k_{i-1}k_{i-3}$, and since $k_i^2 \geq k_{i+1}k_{i-1}$, $(a^2+b^2)^2 k_i^2 \geq (a^2+b^2)^2 k_{i+1}k_{i-1}$ follows.

Now, the previous inequality is a consequence of

$$2(-a^2+b^2)k_ik_{i-2} + (3a^2-b^2)k_{i-1}^2 \geq (a^2+b^2)k_{i+1}k_{i-3}.$$

We can see by Lemma 2.1 that $k_ik_{i-1} \geq k_{i+1}k_{i-2}$ and $k_{i-1}k_{i-2} \geq k_ik_{i-3}$, and because $a \leq 0$,

$$-2a(a^2+b^2)k_ik_{i-1} \geq -2a(a^2+b^2)k_{i+1}k_{i-2} \text{ and } -2ak_{i-1}k_{i-2} \geq -2ak_ik_{i-3}.$$

Finally, we must show that $2(-a^2+b^2)k_ik_{i-2}+(3a^2-b^2)k_{i-1}^2 \geq (a^2+b^2)k_{i+1}k_{i-3}$ is true. Observe that log concavity and Lemma 2.1 imply $k_{i-1}^2 \geq k_ik_{i-2} \geq k_{i+1}k_{i-3}$. Thus we have

$$\begin{array}{rl} & 2(-a^2+b^2)k_ik_{i-2} + (3a^2-b^2)k_{i-1}^2 \\ \geq & 2(-a^2+b^2)k_ik_{i-2} + (3a^2-b^2)k_ik_{i-2} \\ = & (a^2+b^2)k_ik_{i-2} \\ \geq & (a^2+b^2)k_{i+1}k_{i-3}. \end{array}$$

We have thus proven that $\ell_i^2 \geq \ell_{i-1}\ell_{i+1}$ for all $i = 3, ..., n-1$.

Case 4: $\ell_2^2 \geq \ell_1\ell_3$. Expanding and collecting like terms, we must show

$$\begin{array}{rl} & (a^2+b^2)k_2^2 + 3a^2k_1^2 + 2b^2k_0k_2 + k_0^2 \\ \geq & (a^2+b^2)k_1k_3 + b^2k_1^2 + 2a^2k_0k_2 + 2a(a^2+b^2)k_1k_2 - 2a(a^2+b^2)k_0k_3 + 2ak_0k_1. \end{array}$$

We omit the proof of this inequality, for it is precisely parallel to that of Case 2, including the need to consider the subcases $|a| \leq |b|$ and $|a| > |b|$.

Case 5: $\ell_1^2 \geq \ell_0\ell_2$. Expanding and collecting like terms, we must show

$$3a^2k_0^2 + (a^2+b^2)^2k_1^2 \geq b^2k_0^2 + (a^2+b^2)^2k_0k_2 + 2a(a^2+b^2)k_0k_1.$$

We omit the proof of this inequality as well, for it is precisely parallel to the proof of Case 1.

We have now proven the defining condition for log-concavity of p in every case, so our proof is complete.

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